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Superradiance of Blackholes

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The Klein–Gordon equation is separated and the superradiance phenomenon is studied on five-parameter (mass, angular momentum, cosmological constant, electric and magnetic charges) black-hole spacetimes.

1. INTRODUCTION

In recent years there has been considerable interest in the extension of the superradiance phenomenon of black holes to the more general Plebanski (1975) and Plebanski–Demianski (1976) spacetimes. Some physically interesting spacetimes, namely the black-hole spacetimes as well as their generalization by Carter (1973) to include a cosmological term, can be obtained from the Plebanski spacetime by appropriate limiting procedures. Such spacetimes as the Kinnersley (1969), Plebanski (1975), Demianski–Newman (1966), Kerr–Newman (Newman *et al.*, 1965), Brill (1965), Carter (1973), and Bertotti (1959) spacetimes can be obtained from the Plebanski–Demianski spacetime by limiting procedures. In the Plebanski spacetime (Ahmed and Dolan, 1986; Ahmed, 1987, 1988) as well as in the Plebanski–Demianski spacetime (Ahmed and Ansary, 1990) the superradiance phenomenon occurs in cases of boson (gravitational and electromagnetic) fields, whereas no superradiance phenomenon occurs in the cases of fermion (electron and neutrino) fields.

In this paper, we study the superradiance phenomenon for a scalar field in black-hole spacetimes generalized with a cosmological constant as well as magnetic monopole parameters. Spacetimes with a cosmological parameter have attracted renewed interest as models of the inflationary scenario of the early universe. Moreover, interest in spacetimes with a magnetic monopole

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has grown since the development of gauge theories has shed new light on the magnetic monopole, even though the ingenious suggestion by Dirac that the magnetic monopole exists was neglected due to failure to detect such a particle.

For an investigation of superradiance for a scalar field in this spacetime, we separate the Klein–Gordon equation in this spacetime. First we give a brief account of the background spacetime.

2. THE BACKGROUND SPACETIME

We consider the spacetime

$$ds^2 = \Sigma\, \Delta_r^{-1}(dr^2 + \Delta_\theta^{-1}\, d\theta^2) + \Sigma^{-1}(\Delta_\theta \sin^2\theta\, W_1^2 - \Delta_r W_2^2) \qquad (1)$$

where

$$\Sigma = r^2 + a^2 \cos^2\theta$$

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{1}{3}\Lambda r^2\right) - 2Mr + Q^2 + P^2$$

$$\Delta_\theta = 1 + \frac{1}{3}\Lambda a^2 \cos^2\theta$$

$$W_1 = \mathsf{E}^{-1}[a dt - (r^2 + a^2)d\phi]$$

$$W_2 = \mathsf{E}^{-1}(dt - a \sin^2\theta\, d\phi)$$

$$\mathsf{E} = 1 + \frac{1}{3}\Lambda a^2$$

This is a solution to the Einstein–Maxwell equations with nonzero cosmological constant Λ, mass M, angular momentum per unit mass a, electric charge Q, and magnetic charge P.

The electromagnetic vector potential is

$$A_\mu dx^\mu = \Sigma^{-1}(P \cos\theta\, W_1 + QrW_2) \qquad (2)$$

The angular velocities of the event horizon and of the cosmological horizon are

$$\Omega_{\rm H} = \frac{a}{r_{\rm H}^2 + a^2} \quad \text{and} \quad \Omega_C = \frac{a}{r_C^2 + a^2} \qquad (3)$$

The electric potentials of the event horizon and the cosmological horizon are

$$V_{\mathrm{H}} = \frac{Qr_{\mathrm{H}} + aP}{\Xi(r_{\mathrm{H}}^2 + a^2)} \qquad \text{and} \qquad V_C = \frac{Qr_C + aP}{\Xi(r_C^2 + a^2)} \tag{4}$$

where r_{H} and r_C are the respective positions of the event horizon and of the cosmological horizon.

The metric (1) includes as special cases (i) the Kerr–Newman black hole when $P=\Lambda=0$, (ii) the Kerr black hole for $Q=P=\Lambda=0$, (iii) the Kerr–Newman de Sitter black hole if $P=0$, and (iv) the Kerr–de Sitter black hole provided $Q=P=0$.

3. SEPARATION OF THE KLEIN–GORDON EQUATION

The Klein–Gordon equation (Schwinger, 1975) in a general spacetime with a background electromagnetic field is

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial}{\partial x^{\mu}} - ieA_{\mu}\right)\left[\sqrt{-g}g^{\mu\nu}\left(\frac{\partial}{\partial x^{\nu}} - ieA_{\nu}\right)\psi\right] = \mu'^2\psi \tag{5}$$

where $g = |g_{\mu\nu}|$, ψ is the complex scalar field, A is the four-potential of the electromagnetic field, and e and μ' are electric charge and mass of the particle, respectively. Here, for simplicity, we have considered the case in which the magnetic charge of the particle is zero. If we write equation (5) on the background spacetime given by equation (1), the parameters (M, a, P, Q, Λ) become parameters of the field equation.

We shall assume that the components of the wave function in the present context have the dependence $\exp[-i\omega t + i(m - \Xi^{-1}eP)\phi]$ on t and ϕ, where m and $\Xi^{-1}eP$ are integers or half integers.

Therefore, setting

$$\psi = R(r)S(\theta)\Phi(t, \phi) \tag{6}$$

$$\Phi(t, \phi) = \exp[-i\omega t + \mathrm{i}(m - \Xi^{-1}eP)\phi]$$

we can separate equation (5) into the following angular and radial parts:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\Delta_\theta \sin\theta \frac{dS}{d\theta}\right) + \left[\frac{\Xi^2}{\Delta_\theta \sin^2\theta}\{-\omega^2a^2\sin^4\theta + 2\Xi^{-1}eP\omega a \sin^2\theta\cos\theta - (m - \Xi^{-1}eP + \Xi^{-1}eP\cos\theta)^2\} - \mu'^2 a^2\cos^2\theta + \lambda\right]S = 0 \tag{7}$$

$$\frac{d}{dr}\left(\Delta_r \frac{dR}{dr}\right) + VR = 0 \tag{8}$$

where

$$V = \frac{\mathsf{E}^2 K^2}{\Delta_r} - \mu'^2 r^2 - \lambda$$

$$K = \omega(r^2 + a^2) - am + \mathsf{E}^{-1} e(Qr + aP)$$

$$-\lambda = 2a\omega(m - \mathsf{E}^{-1} eP) - a^2\omega^2 - \lambda_1$$

and λ_1 is the separation constant which appears as an eigenvalue in the eigenvalue equation (7). Specializing the parameters (a, M, P, Q, Λ), one arrives at the special subclasses of angular and radial equations. With appropriate limits, the radial equation (8) reduces to that of Teukolsky (1973) in the case of a scalar field. If $\Lambda = 0$, equation (8) corresponds to Semiz's equation (10) (Semiz, 1992).

4. SUPERRADIANCE

Introducing the coordinate r^* defined by

$$\frac{d}{dr^*} = \frac{\Delta_r}{r^2 + a^2}\frac{d}{dr} \tag{9}$$

together with a new function $\hat{R}$ defined by

$$\hat{R} = (r^2 + a^2)^{1/2} R \tag{10}$$

we can reduce equation (8) to the following compact form:

$$\frac{d^2\hat{R}}{dr^{*2}} + \left[\frac{\mathsf{E}^2 K^2}{(r^2 + a^2)^2} - \frac{\Delta_r}{r^2 + a^2} \times \left\{\mu'^2 r^2 + \lambda + \sqrt{(r^2 + a^2)}\,\frac{d}{dr}\left\{\frac{r\Delta_r}{(r^2 + a^2)^{3/2}}\right\}\right\}\right]\hat{R} = 0 \tag{11}$$

As $r^* \to +\infty$ equation (11) becomes

$$\frac{d^2\hat{R}}{dr^{*2}} + \mathsf{E}^2(\omega - \omega_+)^2\hat{R} = 0 \tag{12}$$

where

$$\omega_+ = m\Omega_c + eV_C$$

and as $r^* \to -\infty$, equation (11) becomes

$$\frac{d^2\hat{R}}{dr^{*2}} + \mathsf{E}^2(\omega - \omega_-)^2\hat{R} = 0 \tag{13}$$

where

$$\omega_- = m\Omega_H + eV_H$$

Now we impose the boundary condition such that the group velocity of the wave for $r^* \to -\infty$ is directed toward the event horizon. That is, the wave plunges in from the cosmological horizon, partially passes through the potential barrier, and falls through the event horizon while the rest reflects back to the cosmological horizon. Turning to the problem of reflection and transmission, we seek solutions for $\hat{R}$ which have the asymptotic behaviors

$$\left.\begin{array}{ll} & \hat{R} \to B\exp\{-i(\omega - \omega_-)r^*\}, \\ r^* \to -\infty & \hat{R} \to \exp\{-i(\omega - \omega_+)\}r^* + A\exp\{i(\omega - \omega_+)\}r^*, \\ r^* \to +\infty & \end{array}\right\}$$

For solutions having these asymptotic behaviors,

$$\left(\hat{R}\,\frac{d\hat{R}^*}{dr^*} - \hat{R}^*\,\frac{d\hat{R}}{dr^*}\right)_{r^* \to -\infty} = 2i(\omega - \omega_-)|B|^2 \tag{15}$$

and

$$\left(\hat{R}\,\frac{d\hat{R}^*}{dr^*} - \hat{R}^*\,\frac{d\hat{R}}{dr^*}\right)_{r^* \to +\infty} = 2i(\omega - \omega_+)(1 - |A|^2) \tag{16}$$

Hence, by the constancy of the Wronskian of $\hat{R}$ in equation (11), we find the following relation:

$$1 - |A|^2 = \left(\frac{\omega - \omega_-}{\omega - \omega_+}\right)|B|^2 \tag{17}$$

Equation (17) implies that if $(\omega - \omega_-)/(\omega - \omega_+) < 0$, $|A|^2 > 1$, i.e., the amplitude of the reflected wave is greater than that of the incident wave. This is the phenomenon of superradiance, and the condition for the existence of superradiance is $\omega_+ < \omega < \omega_-$.

5. REMARKS

This work can be extended easily to spacetimes like those of Plebanski and Plebanski–Demianski which include all the black-hole solutions which are asymptotically flat as well as asymptotically de Sitter and also other interesting spacetimes.

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